5.5 Bases other than e

Warm-up:

① $y = (4x^5+3)e^{4x^4} \rightarrow \boxed{20x^4(e^{4x^4}) + (4x^5+3)(e^{4x^4}(16x^3))}$

② $\int 60x^3 e^{3x^4-2}\,dx$ $\quad u = 3x^4-2 \quad du = 12x^3\,dx$

$\rightarrow 5\int e^u\,du \rightarrow 5[e^u] + C \rightarrow \boxed{5e^{3x^4-2} + C}$

③ $\int -20\csc^2 4x \cdot e^{\cot 4x}\,dx \quad u = \cot 4x \quad du = -4\csc^2(4x)$

$5\int -4\csc^2 4x \cdot e^u \rightarrow 5\int e^u\,du \rightarrow \boxed{5e^{\cot 4x} + C}$

Exponential Function to Base "a"

If "a" is positive, $(a \neq 1)$ and x is real, exponential function to the base "a" is a^x

$a^x = e^{(\ln a)x} \qquad b^{(\log_b(u))} = u$

Change of base formula

$\log_b a = \dfrac{\log_{\text{desired base}} a}{\log_{\text{desired base}} b}$

desired base = e or 10

b = starting base

E.g.: $\log_3 4 = \dfrac{\ln 4}{\ln 3} = \dfrac{\log_{10} 4}{\log_{10} 3} = \dfrac{\log_{13} 4}{\log_{13} 3}$

Ex 1: Radioactive half-life

model: $y = \underset{\text{sample}}{a} \cdot \left(\frac{1}{2}\right)^{t/\text{half life}}$ (t = years)

$\rightarrow 1 \cdot \left(\frac{1}{2}\right)^{\left(\frac{10000}{5715}\right)} \rightarrow 0.297\,g$

1g carbon-14 after 10,000 years

Log function to base "a"

$\log_a x = \dfrac{1}{\ln a} \ln x \qquad \log_a x = \dfrac{\ln x}{\ln a} = \dfrac{1}{\ln a} \cdot \ln x$

Properties of Logs

1. $\log_a 1 = 0$
2. $\log_a xy = \log_a x + \log_a y$
3. $\log_a x^n = n \log_a x$
4. $\log_a \frac{x}{y} = \log_a x - \log_a y$

Memorize

Properties of inverse functions

1. $y = a^x$ iff $x = \log_a y$
2. $a^{\log_a x} = x$, for $x > 0$
3. $\log_a a^x = x$

Ex 2:

(a) $3^x = \frac{1}{81} \to 3^x = \frac{1}{3^4} \to 3^x = 3^{-4} \to x = -4$

(b) $\log_2 x = -4 \to 2^{-4} = x \to \frac{1}{16} = x$

$\frac{du}{dx} = u'$

Derivatives for bases other than e

1. $\frac{d}{dx}[a^x] = (\ln a)a^x$
2. $\frac{d}{dx}[a^u] = (\ln a)a^u \frac{du}{dx}$
3. $\frac{d}{dx}[\log_a x] = \frac{1}{(\ln a)x}$
4. $\frac{d}{dx}[\log_a u] = \frac{1}{(\ln a)u} \frac{du}{dx} = \frac{u'}{u \ln a}$

Ex 3: Derive

(a) $y = 2^x$ $\quad a = 2 \to \boxed{\frac{dy}{dx} = (\ln 2)2^x}$

(b) $y = 2^{3x}$ $\quad u = 3x \quad u' = 3 \to \boxed{\frac{dy}{dx} = (\ln 2)(2^{3x})(3)}$

ⓒ $y = \log_{10} \cos x \rightarrow \frac{u'}{u \ln a}$

$\frac{dy}{dx} = \frac{-\sin x}{\cos x \ln 10} \rightarrow \boxed{\frac{-\tan x}{\ln 10}}$

ⓓ $y = \log_3 \frac{\sqrt{x}}{x+5} \rightarrow \frac{1}{2}\log_3 x - \log_3(x+5)$

$\rightarrow y' = \left(\frac{1}{2}\left(\frac{1}{x \ln 3}\right)\right) - \frac{1}{(x+5)\ln 3} \rightarrow \boxed{\frac{1}{2\ln 3\, x} - \frac{1}{(x+5)\ln 3}}$

<u>Integrals of Bases other than "e"</u>

$\int a^x dx = \left(\frac{1}{\ln(a)}\right) a^x + C \rightarrow \int a^u du = \frac{1}{\ln a} a^u + C$

<u>Ex 4:</u> ⓐ $\int 2^x dx \rightarrow \left(\frac{1}{\ln 2}\right) 2^x + C = \boxed{\frac{2^x}{\ln 2} + C}$

ⓑ $\int 3^{4x} dx \quad u = 4x \quad du = 4\, dx$

$\rightarrow \frac{1}{4}\int 3^u du \rightarrow \frac{1}{4}\frac{1}{\ln 3}(3^u) + C$

$\rightarrow \frac{1}{4}\frac{1}{\ln 3}(3^{4x}) + C \rightarrow \boxed{\frac{3^{4x}}{4\ln 3} + C}$

ⓒ $\int \sin x (2^{\cos x})\, dx \quad u = \cos x \quad du = -\sin x\, dx$

$-\int 2^u du \rightarrow -\left(\frac{1}{\ln 2}\right)(2^{\cos x}) + C \rightarrow \boxed{\frac{-2^{\cos x}}{\ln 2} + C}$

<u>Ex 5:</u>

ⓐ $\frac{d}{dx}[e^e] = 0$ ⓑ $\frac{d}{dx}[e^x] = e^x$ ⓒ $\frac{d}{dx}[x^e] = e x^{(e-1)}$

ⓓ $\frac{d}{dx}[x^x] \Rightarrow \ln y = \ln x^x \rightarrow y' = x \ln x \rightarrow \frac{y'}{y} = x\left(\frac{1}{x}\right) + 1\ln x$

$\rightarrow y' = y(1 + \ln x) \rightarrow y' = x^x(1 + \ln x)$

Compounded n times/year: $A = P\left(1+\frac{r}{n}\right)^{nt}$

Compounded continuously: $A = Pe^{rt}$

<u>Ex 6:</u> $n=4$ $P=2500$ 5%

$n=12$

$n=4$ $A = 2500\left(1+\frac{0.05}{4}\right)^{4(5)} = 3205.09$

$n=12$ $A = 2500\left(1+\frac{0.05}{12}\right)^{(12(5))} = 3208.40$

continuously: $A = 2500e^{0.05(5)} = 3210.06$

5.6 Inverse Trig Functions

Warm-up: 1) $y = x^3 3^{2x}$ $\qquad 3x^2(u') + (x^3)(u)$

$\to u' = (\ln 3)\, 3^{2x}(2) \to 3^{2x}(\ln 3)(2)$

$\to 3x^2\left(3^{2x}(\ln 3)2\right) + x^3\left(3^{2x}\right)$

$\to 3^{2x}x^2\left(6(\ln 3) + x\right)$

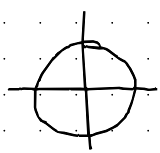
2) $f(x) = \log_5(3x+4) \to \dfrac{3}{(3x+4)\ln 5} \to \dfrac{3}{\ln 5\,(3x+4)}$

Inverse Trig Functions

		D	R
$y = \sin^{-1} x$	iff $\sin y = x$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$y = \cos^{-1} x$	iff $\cos y = x$	$[-1, 1]$	$[0, \pi]$
$y = \tan^{-1} x$	iff $\tan y = x$	$(-\infty, \infty)$	$(-\frac{\pi}{2}, \frac{\pi}{2})$

Ex 1:

a) $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$



b) $\cos^{-1}(0) = \frac{\pi}{2}$

c) $\tan^{-1}\left(\dfrac{\sqrt{3}/2}{1/2}\right) \frac{y}{x} \to \frac{\pi}{3}$

$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

d) $\sin^{-1}(0.3) \to 0.305$

Ex 2:

$\tan(\arctan(2x-3)) = \tan\left(\frac{\pi}{4}\right) \to 2x - 3 = \tan\left(\frac{\pi}{4}\right) \to 2x - 3 = 1 \to x = 2$

Ex 3: a) $\cos\left(\cos^{-1}\left(\frac{1}{5}\right)\right) \to \frac{1}{5}$ $\qquad$ e) $\tan^{-1}\left(\tan \frac{4\pi}{3}\right) \to \frac{\pi}{3}$

b) $\tan\left(\tan^{-1}(5.2)\right) \to 5.2$ $\qquad$ d) $\sin^{-1}\left(\sin \frac{5\pi}{6}\right) \to \frac{\pi}{6}$

c) $\sin\left(\sin^{-1}\left(3/2\right)\right) \to$ undefined

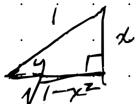
d) $\cos^{-1}\left(\cos \frac{\pi}{4}\right) \to \frac{\pi}{4}$

sin → reflect y-axis
cos → reflect x-axis
tan → reflect origin

Ex 4:

ⓐ $y=\sin^{-1}x,\ 0<y<\frac{\pi}{2}$ find $\cos y$

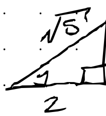
$\rightarrow \sin^{-1}\frac{x}{1}\ \frac{\text{opp}}{\text{hyp}}$



$\cos y = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$

ⓑ $y=\text{arcsec}\left(\frac{\sqrt{5}}{2}\right)\ \frac{\text{Hyp}}{\text{Adj}}$ find $\tan y$

$\sec\theta = \frac{\text{Hyp}}{\text{Adj}}\quad \cos\theta = \frac{\text{Adj}}{\text{Hyp}}$



$\sqrt{5-4} \rightarrow \sqrt{1} \rightarrow 1$

$\tan y \rightarrow \frac{1}{2}$

Derivatives of Inverse Trig funcs.

"co-"'s just negative

$$\frac{d}{dx}\left[\sin^{-1}u\right] = \frac{u'}{\sqrt{1-u^2}} \rightarrow \frac{d}{dx}\left[\cos^{-1}u\right] = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}\left[\tan^{-1}u\right] = \frac{u'}{1+u^2} \rightarrow \frac{d}{dx}\left[\cot^{-1}u\right] = \frac{-u'}{1+u^2}$$

$$\frac{d}{dx}\left[\sec^{-1}u\right] = \frac{u'}{|u|\sqrt{u^2-1}} \rightarrow \frac{d}{dx}\left[\csc^{-1}u\right] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

Ex 4: ⓐ $\sin^{-1}(2x) \rightarrow \frac{2}{\sqrt{1-(2x)^2}} \rightarrow \frac{2}{\sqrt{1-4x^2}}$

ⓑ $\tan^{-1}(3x) \rightarrow \frac{3}{1+(3x)^2} \rightarrow \frac{3}{1+9x^2}$

ⓒ $\sin^{-1}(\sqrt{x}) \rightarrow \frac{\frac{1}{2\sqrt{x}}}{\sqrt{1-x}} \rightarrow \frac{1}{2\sqrt{x}} \cdot \frac{1}{\sqrt{1-x}} \rightarrow \frac{1}{2\sqrt{x-x^2}}$

ⓓ $\sec^{-1}(e^{2x}) \rightarrow \frac{2e^{2x}}{|e^{2x}|\sqrt{(e^{2x})^2-1}} \rightarrow \frac{2e^{2x}}{e^{2x}\sqrt{e^{4x}-1}} \rightarrow \frac{2}{\sqrt{e^{4x}-1}}$

Ex 5:

$$y = \sin^{-1} x + x\sqrt{1-x^2}$$

$$\to \frac{1}{\sqrt{1-x^2}} + x\left(\frac{1}{2}(1-x^2)^{-1/2}(-2x)\right) + 1(1-x^2)^{1/2}$$

$$\to \frac{1}{\sqrt{1-x^2}} + \frac{-x^2}{\sqrt{1-x^2}} + \frac{\sqrt{1-x^2}}{1}\frac{(\sqrt{1-x^2})}{(\sqrt{1-x^2})}$$

$$\to \frac{1-x^2+1-x^2}{\sqrt{1-x^2}} \to 2-2x^2 \to \frac{2(1-x^2)}{\sqrt{1-x^2}} \to 2\sqrt{1-x^2}$$

5.7 Inverse Trig. Function Integration

Warm up: $12x^2$

1. $y = \sec^{-1} 4x^3 \to \frac{1}{|u|\sqrt{u^2-1}} \to \frac{12x^2}{|4x^3|\sqrt{16x^6-1}}$

2. $y = \cot^{-1} 2x^4$ $\quad 8x^3$ $\quad \frac{-8x^3}{4x^8+1}$

3. $y = \cos^{-1} 2x^2$ $\quad 4x$ $\quad \frac{-4x}{\sqrt{1-4x^4}}$

Integrals Involving Inverse Trig. Functions

1. $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin \frac{u}{a} + C$

2. $\int \frac{du}{a^2+u^2} = \frac{1}{a} \arctan \frac{u}{a} + C$

3. $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{|u|}{a} + C$

"co-"s are these formulae but negative

Ex 1:

$u = x \quad du = 1\,dx \quad a = 2$

(a) $\int \frac{1}{\sqrt{4-x^2}}\,dx$ arcsin?

$\to \boxed{\arcsin \frac{x}{2} + C}$

$u = 3x \quad du = 3\,dx \quad a = \sqrt{2}$

(b) $\frac{1}{3}\int \frac{1\,(3)}{2+9x^2}\,dx$ arctan?

$\frac{1}{3}\left(\frac{1}{\sqrt{2}} \arctan \frac{3x}{\sqrt{2}} + C\right) \to \frac{1}{3\sqrt{2}} \arctan \frac{3x}{\sqrt{2}} + C$

$u = 2x \quad du = 2\,dx \quad a = 3$

(c) $\int \frac{1\,(2)}{(2)x\sqrt{4x^2-9}}$ arcsec?

$= \frac{1}{3} \operatorname{arcsec} \frac{|2x|}{3} + C$

Ex 2:

$$\int \frac{1}{(e^x)\sqrt{e^{2x}-1}}(e^x) \quad u=e^x \quad du=e^x dx \quad a=1$$

arcsec? $\frac{1}{u\sqrt{u^2-a^2}}du$

$\rightarrow \frac{1}{1}\text{arcsec}\frac{|e^x|}{1}+C \rightarrow \text{arcsec}|e^x|+C$

Ex 3: $\int \frac{x+2}{\sqrt{4-x^2}}dx$ arc sin $u=x \quad du=dx \quad a=2$

$\rightarrow \int \frac{x}{\sqrt{4-x^2}}dx + 2\int \frac{1}{\sqrt{4-x^2}}dx$ (u-sub) (arcsin)

$\rightarrow u=4-x^2 \quad du=-2x\,dx$

$-\frac{1}{2}\int \frac{1}{u^{1/2}}du \rightarrow -\frac{1}{2}\int u^{-1/2}du \rightarrow -\frac{1}{2}\left[\frac{2u^{1/2}}{1}\right]+C$

$\rightarrow -\sqrt{4-x^2}+C$

$\rightarrow u=x \quad du=dx \quad a=2$ coefficient $=2$

$+2\left(\arcsin\frac{x}{2}\right)+C$

$\rightarrow -\sqrt{4-x^2}+2\arcsin\frac{x}{2}+C$

Ex 4: $\int \frac{1}{x^2-4x+7}dx$ arctan? Use "completing the square"

$x^2-4x+7 \rightarrow (x^2-4x+4)+7-4 \quad \left(\frac{b}{2}\right)^2$

$\hookrightarrow (x-2)^2+3$

$u^2+a^2 \quad u=x-2 \quad du=dx \quad a=\sqrt{3}$

$\rightarrow \int \frac{1}{(x-2)^2+3}dx$ arctan

$\frac{1}{\sqrt{3}}\arctan\frac{x-2}{\sqrt{3}}+C$

Ex 5: $\displaystyle\int_{3/2}^{9/4} \frac{1}{\sqrt{3x - x^2}}$ arcsin

Complete the Square $\left(\frac{b}{2}\right)^2$ $\left(-\frac{3}{2}\right)^2$

$\to \displaystyle\int \frac{1}{\sqrt{9/4 - (x - \frac{3}{2})^2}}\,dx$

$3x - x^2$

$\to -x^2 + 3x$

$-\left(x^2 - 3x + \frac{9}{4}\right) + \frac{9}{4}$

$= \arcsin \dfrac{x - 3/2}{3/2} + C$

$\to \left[\arcsin \dfrac{2x - 3}{3} + C\right]$ $\dfrac{9}{4} - \left(x - \dfrac{3}{2}\right)^2$

indef.

$a^2 - u^2 \to \arcsin$

$\to \left[\arcsin \dfrac{2x - 3}{3}\right]_{3/2}^{9/4} = \left[\arcsin \frac{1}{2} - \arcsin 0\right]$

$\left(\frac{9}{4} - \frac{3}{2}\right)\frac{2}{3} \to \left(\frac{3}{4}\right)\frac{2}{3} \to \frac{6}{12} \to \frac{1}{2}$

$\to \left[\frac{\pi}{6} - 0\right] \to \frac{\pi}{6}$

8.7 Indeterminate Forms & L'Hopital's Rule

Warm-up:

$$\int \frac{\sin^2 x - \cos^2 x}{\cos x}\,dx \to \int \frac{1-\cos^2 x - \cos^2 x}{\cos x}\,dx$$

$$\to \int \frac{1-2\cos^2 x}{\cos x}\,dx \to \int (\sec x - 2\cos x)\,dx$$

$$\to \boxed{\ln|\sec x + \tan x| - 2\sin x\ dx}$$

---

1. $\lim\limits_{x\to 1} \frac{4x}{x^2+4} \to \boxed{\frac{4}{5}}$

2. $\lim\limits_{x\to 2} \frac{-x-2}{x^2-4} \to \lim\limits_{x\to 2} \frac{-(\cancel{x-2})}{(\cancel{x-2})(x+2)} \to \boxed{\frac{-1}{4}}$

3. $\lim\limits_{x\to 3} \frac{x-3}{\sqrt{x+6}-3}\left(\frac{\sqrt{x+6}+3}{\sqrt{x+6}+3}\right)$

$\to \lim\limits_{x\to 3} \frac{(\cancel{x-3})(\sqrt{x+6}+3)}{\cancel{x+6-9}} \to \cancel{x-3}$

$\to \sqrt{9}+3 \to \boxed{6}$

Expand every "ln" before deriving

Derivative of Inverse of function

Derivatives for ~~bases~~ other than e

Completing the square

Quiz Stuff (Ch 5)

Indeterminate Forms:

1. Try direct ~~sub~~stitution, you end up with $\frac{0}{0}$
2. " " " , you get $\frac{\pm\infty}{\pm\infty}$

L'Hôpital's Rule - Use on indeterminate limits

If $\lim\limits_{x\to c} \frac{f(x)}{g(x)} = \frac{0}{0}$, then $\lim\limits_{x\to c} \frac{f(x)}{g(x)} = \lim\limits_{x\to c} \frac{f'(x)}{g'(x)}$,

repeat if you end up with $\frac{0}{0}$ again, a few times. Then Indeterminate

Test: Must write L'H when doing L'Hopital's Rule

Ex 1: $\lim\limits_{x\to 0} \frac{e^{2x}-1}{x} \to \frac{0}{0} \to$ L'Hôptial's Rule

$\to (e^{2x})(2) \to \frac{2e^{2x}}{1} \to \lim\limits_{x\to 0} 2e^{2x} \to 2e^0 \to \boxed{2}$

Ex 2: $\lim_{x\to\infty} \frac{\ln x}{x} \to \frac{\ln(\infty)}{\infty} \to \frac{\infty}{\infty}$ L'Hopital's

$\to \frac{\frac{1}{x}}{1} \to \frac{1}{x} \to \lim_{x\to\infty} \frac{1}{x} \to \frac{1}{\infty} = \boxed{0}$

Ex 3:

$\lim_{x\to-\infty} \frac{x^2}{e^{-x}} = \frac{\infty}{\infty}$ L'Hopital's Rule

$\to \lim_{x\to-\infty} \frac{2x}{-e^{-x}} \to \frac{-\infty}{-\infty}$ L'H $\to \lim_{x\to-\infty} \frac{2}{e^{-x}} = \frac{2}{\infty}$ $\to \boxed{0}$

Ex 4: $\lim_{x\to\infty} e^{-x}\sqrt{x} \to 0\sqrt{\infty} \to 0\cdot\infty$ L'H

$\to \lim_{x\to\infty} \frac{\sqrt{x}}{e^{x}} \to \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{e^{x}} \to \frac{0}{\infty} \longrightarrow \boxed{0}$

6.1 Slope Fields

③ $\lim_{x\to 0}\frac{e^x-1}{4x^3} = \frac{e^0-1}{4\cdot 0^3} = \frac{0}{0}$

L'H $\lim_{x\to 0}\frac{e^x}{12x^2} \to \frac{1}{0} \to$ DNE / ∞

② $\lim_{x\to 1}\frac{(x-1)}{\ln x} \to \frac{1-1}{\ln 1} = \frac{0}{0}$

L'H $\lim_{x\to 1}\frac{1}{1/x} = \frac{x}{1} = \frac{1}{1} \to 1$

③ $\lim_{x\to 0}\frac{2(e^x-1-x)}{x^2} \to \frac{0}{0}$

L'H $\lim_{x\to 0}\frac{2e^x-2-2x}{x^2} = \frac{2e^x-2}{2x} = \frac{2e^x}{2} \to \frac{2}{2} \to 1$

④ $\lim_{x\to 0}\frac{2(e^x-x)}{x^2}$ L'H $\frac{2e^x-2x}{x^2} \to \frac{2}{0}$ — DNE/∞

$\frac{0}{0}$

General Solution → solution with a C

Specific / Particular Solution → uses given point to find C

Ex1: $y''-y=0$ } Prove whether (a, b, c) are solutions of this differential solution

ⓐ $y=\sin x \to -\sin x - \sin x \neq 0$

$\to y' = \cos x$ $\quad y=\sin(x)$ is not a solution

$\to y'' = -\sin x$

$y''-y=0$

ⓑ $y=4e^{-x}$ $\quad y''-y=0$

$y'=-4e^{-x}$ $\quad 4e^{-x}-4e^{-x}=0$

$y''=4e^{-x}$ $\quad y=4e^{-x}$ is a solution

ⓒ $y=Ce^x$

$y'=Ce^x$ $\quad y''-y=0 \to Ce^x-Ce^x=0$

$y''=Ce^x$ $\quad y=Ce^x$ is a solution

Ex 2:

$xy'-3y=0$ verify Cx^3 is a solution. Then find particular solution by initial conditions $y=2$ & $x=-3$

$y'=3Cx^2$ $\quad xy'-3y=0$

$x(3Cx^2)-3(Cx^3) \to 3Cx^3-3Cx^3=0$ ✓

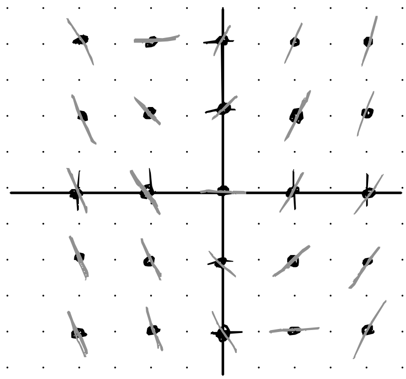
$y=Cx^3$ $\quad 2=C(-3)^3$ $\quad 2=C(-27)$

$\frac{2}{-27}=C$ partic. soln.: $y=\frac{-2}{27}x^3$

Ex 3: Sketch slope field for the differential equation

$y' = 2x + y$ use the slope field to graph solution @ (1, 1)

x	y	y'
0	0	0
1	1	3
2	2	6
1	-2	0
0	1	1
0	2	2
0	-1	-1 $-1/1$
0	-2	-2 $-2/1$
1	-1	1
1	0	2
1	2	4



6.2 Growth & Decay

Ex 1: $y' = \frac{2x}{y}$

$$\frac{dy}{dx} = \frac{2x}{y}$$

$$dy = \frac{2x}{y}\,dx$$

$$y\,dy = 2x\,dx$$

$$\int y\,dy = \int 2x\,dx$$

$$\rightarrow \frac{y^2}{2} = x^{2+C} \Rightarrow y^2 = 2x^{2+C} \Rightarrow y = \pm\sqrt{2x^2 + C}$$

Warm up:

1. $\lim_{x\to 0} \frac{1-\cos(3x)}{\cos(5x)-1} \rightarrow \frac{1-\cos(0)}{\cos(0)-1} \Rightarrow \frac{0}{0}$

L'H $\lim_{x\to 0} \frac{3\sin(3x)}{-5\sin 5x} \rightarrow \frac{3(0)}{5(0)} \rightarrow \frac{0}{0}$

L'H $\lim_{x\to 0} \frac{9\cos 3x}{-25\cos 5x} \rightarrow \frac{9(1)}{-25(1)} \rightarrow \boxed{-\frac{9}{25}}$

2. $\int \frac{e^x}{e^x+1}\,dx \Rightarrow y = \ln(e^x+1) + \ln 3$

$\frac{dy}{dx} = \frac{e^x}{e^x+1}$ $\quad u = e^x+1$, $du = e^x\,dx$ $\quad y(0) = \ln 6$

$\int dy = \int \frac{e^x}{e^x+1}\,dx$

$y = \int \frac{1}{u}\,du \Rightarrow y = \ln|u| + C$

$\rightarrow y = \ln|e^x+1| + C$

$\ln 6 = \ln|e^0+1| + C \rightarrow \ln 6 = \ln 2 + C$

$\boxed{\ln\frac{6}{2} = C}$

Exponential Growth & Decay Model

If y is diff'able of t such that $y > 0$ & $y' = ky$ for some constant k, then:

$$y = Ce^{kt}$$

C is initial value of y, "k" is the proportionality constant

Rate of Change of $y \longrightarrow \frac{dy}{dt} \overset{\text{is}}{=} ky$ proportional to y

$$dt\left(\frac{dy}{dt}\right) = (ky)\,dy \Rightarrow \frac{dy}{y} = \frac{(ky)\,dt}{y}$$

$$\Rightarrow \frac{dy}{y} = k\,dt$$

$$\rightarrow \int \frac{1}{y}\,dy = \int k\,dt \Rightarrow \ln|y| = kt + C$$

$$\rightarrow e^{\ln|y|} = e^{(kt+C)}$$

$$\rightarrow y = e^{kt} \cdot \underbrace{e^{c}}_{=C} \rightarrow y = Ce^{kt}$$

If "Rate of change of y is proportional to y" is in the question, follow last process (seperation of variables)

Ex 2: $\frac{dy}{dt} = ky \Rightarrow \frac{1}{y}\,dy = k\,dt$

$\rightarrow \int \frac{1}{y}\,dy = \int k\,dt \Rightarrow \ln|y| = kt + c$

$\rightarrow e^{\ln|y|} = e^{(kt+c)} \Rightarrow y = e^{kt} \cdot e^{c}$

$\rightarrow y = Ce^{kt}$

Find C & k

when $t = 0$, $y = 2$ $\quad 2 = Ce^{k(0)} \Rightarrow 2 = C$

$C = 2$ $\quad$ find k $\quad 4 = 2e^{k(2)} \rightarrow 2 = e^{k(2)}$

$\rightarrow \ln 2 = 2k \ln e = 2k$

$\rightarrow \frac{\ln 2}{2} = k$

$\rightarrow y = 2e^{\left(\frac{\ln 2}{2}\right)t} \Rightarrow y(3) = 2e^{\frac{\ln 2}{2}(3)} = 5.657$

Ex 3: 10g plutonium isotope ^{239}Pu

Half life is 24,100 years $\quad$ How long to decay to 1g

$y = Ce^{kt}$ $\quad$ find k $\quad \frac{1}{2} = e^{k(24,100)}$

$\rightarrow \ln\left(\frac{1}{2}\right) = k\,24,100 \Rightarrow k = \frac{\ln\left(\frac{1}{2}\right)}{24,100}$

$1 = 10e^{\frac{\ln \frac{1}{2}}{24,100}(t)} \rightarrow \frac{1}{10} = e^{\frac{\ln \frac{1}{2}}{24100}t}$

$\ln \frac{1}{10} = \frac{\ln \frac{1}{2}}{24100}t \Rightarrow \frac{\ln \frac{1}{10}}{\frac{\ln \frac{1}{2}}{24100}} = t$ $\Rightarrow$ It takes 80,058.467 years for 10g to become 1g.

6.3 Seperation of Variables

Warm up:

$\frac{dy}{dx} = xy \to \int \frac{1}{y}\,dy = \int x\,dx \to \ln|y| = \frac{x^2}{2} + C$

$y = e^{\frac{x^2}{2}+C} \to y = Ce^{\frac{x^2}{2}} \to \boxed{\frac{1}{2} = C}\ \boxed{y = \frac{1}{2}e^{\frac{x^2}{2}}}$

Always +C !!!

Ex 1:

$(x^2+4)\frac{dy}{dx} = xy$

$\int \frac{1}{y}\,dy = \int \frac{x}{x^2+4}\,dx \quad u = x^2+4 \quad du = 2x\,dx$

$\to \ln|y| = \frac{1}{2}\ln|x^2+4| + C$

$e^{(\ln|y|)} = e^{(\ln\sqrt{x^2+4}+C)} \to \boxed{y = C\sqrt{x^2+4}}$

Ex 2: $(1,3)$; $\frac{dy}{dx} = \frac{y}{x^2}$

$\to \int \frac{1}{y}\,dy = \int \frac{1}{x^2}\,dx \to \ln|y| = \int x^{-2}\,dx + C$

$\to \ln|y| = -1x^{-1} + C = \frac{-1}{x} + C$

$y = Ce^{\frac{-1}{x}} \to 3 = Ce^{-\frac{1}{1}} \to 3 = \frac{C}{e}$

$\to 3e = C \to y = 3e(e^{-\frac{1}{x}}) = \boxed{y = 3e^{1-\frac{1}{x}}}$

Ex 3: $\frac{dy}{dx} = x^4(y-2)$ +1pt AP FRQ

$\int \frac{1}{y-2}\,dy = \int x^4\,dx$ +1pt $\to \ln|y-2| = \frac{x^5}{5} + C$ +1pt +1pt

Initial condition: $(0,0)$

$y-2 = Ce^{\frac{x^5}{5}} \to y = Ce^{\frac{x^5}{5}} + 2 \to 0 = Ce^0 + 2$

$C = -2$ +1pt $\to \boxed{y = -2e^{\frac{x^5}{5}} + 2}$ +1pt

6/6

Chapter 5 & 6 Quiz Review

Always round to 3 decimal places: 0.000

(4) $\int_{\ln 4}^{\ln 7} e^{-x}\,dx$

$u = -x$
$du = -dx$
$u' = -1$

$\to -\int e^{u}\,du \to \left[-e^{-x}\right]_{\ln 4}^{\ln 7}$

$\to \left[\frac{-1}{e^{x}}\right]_{\ln 4}^{\ln 7} \to \left[\frac{-1}{e^{\ln 7}} - \frac{-1}{e^{\ln 4}}\right] \to -\frac{1}{7} + \frac{1}{4}$

$\to \frac{-4}{28} + \frac{7}{28} \to \boxed{\frac{3}{28}}$

(2) $f(x)$ & $g(x)$ are inverse if $f(x) = 4x^5 + 4x + 2$

$f'(x) = 20x^4 + 4$

find $g'(10)$:

	x	y
g	10	1
f	1	10

$10 = 4x^5 + 4x + 2 \to 0 = 4x^5 + 4x - 8$

$\to 0 = 4(x^5 + x - 2)$ $\quad \frac{p}{q} = \frac{\pm 1, \pm 2}{\pm 1}$

$$\begin{array}{r|rrrrrr} 1 & 1 & 0 & 0 & 0 & 1 & -2 \\ & & 1 & 1 & 1 & 1 & 2 \\ \hline & 1 & 1 & 1 & 1 & 2 & 0 \end{array}$$

$\to g'(10) = \frac{1}{f'(g(10))} \to \frac{1}{f'(1)} = \frac{1}{20+4} \to \boxed{\frac{1}{24}}$

(3) $y' = \frac{8y}{9x} \to y' = \frac{8}{9} \cdot \frac{y}{x} \to \int \frac{1}{y}\,dy = \int \frac{8}{9}\frac{1}{x}\,dx$

$\to \ln|y| = \frac{8}{9}\ln|x| + C \to \ln y = \ln x^{8/9} + C$

$y = Cx^{8/9} \to @(7,9) \to 9 = C(7)^{8/9}$

$\to C = \frac{9}{7^{8/9}} \to \boxed{y = \left(\frac{9}{7^{8/9}}\right)x^{8/9}}$

(12) $\int \frac{2x-4}{x^2-6x+45}\,dx$ $\quad u=x^2-6x+45 \quad du=(2x-6)dx$

$$\int \frac{2x-6}{x^2-6x+45}\,dx + \int \frac{2}{x^2-6x+45}\,dx$$

$$\rightarrow \int \frac{1}{u}\,du + 2\int \frac{1}{x^2-6x+45}\,dx$$ $\quad u=x-3$, $a=6$

$$\rightarrow \ln|x^2-6x+45| + 2\int \frac{1}{(x-3)^2+36} + C$$

$$\rightarrow \boxed{\ln|x^2-6x+45| + 2\left(\frac{1}{6}\right)\arctan\left(\frac{x-3}{6}\right) + C}$$

(5) $\int e^{4x}\sec(e^{4x})\tan(e^{4x})\,dx$

$u=e^{4x}$
$du=4e^{4x}$ $\quad \rightarrow \frac{1}{4}\int \sec u \tan u\,du$

$$\rightarrow \frac{1}{4}\sec u + C \rightarrow \boxed{\frac{1}{4}\sec(e^{4x}) + C}$$

(13) $\int \frac{1}{x\sqrt{x^4-4}}\,dx$ $\quad u=x^2 \quad du=2x\,dx \quad a=2$

$$\rightarrow \frac{1}{2}\int \frac{2x}{x(x)\sqrt{x^4-4}}\,dx \rightarrow \frac{1}{2}\int \frac{1}{u\sqrt{u^2-a^2}}\,du$$

$$\frac{1}{2}\left(\frac{1}{2}\right)\sec^{-1}\frac{|x^2|}{2} + C \rightarrow \boxed{\frac{1}{4}\sec^{-1}\frac{|x^2|}{2} + C}$$

(6) $f'(x)$ if $f(x)=(x^8)(2^{7x})$ $\quad \frac{d}{dx}[a^u] \rightarrow \ln a\, a^u u'$

$$\rightarrow x^8(\ln 2\,(2^{7x})\,7) + 8x^7(2^{7x})$$

(10) $\int \frac{x}{x^4+9}\,dx$ $\quad \rightarrow u=x^2 \quad a=3 \quad du=2x\,dx$

$$\frac{1}{2}\int \frac{1}{u^2+a^2}\,du \rightarrow \boxed{\frac{1}{2}\left(\frac{1}{3}\arctan^{-1}\frac{x^2}{3}\right) + C}$$

(11) $\int_0^{1/6} \frac{3}{\sqrt{1-9x^2}}\,dx$ $\quad u=3x \quad du=3\,dx \quad a=1$

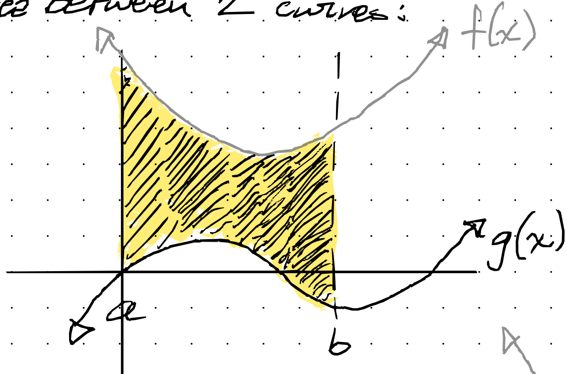
$$\rightarrow \left[\arcsin\frac{3x}{1}\right]_0^{1/6} \rightarrow \left[\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0)\right] \rightarrow \frac{\pi}{6} - 0 \rightarrow \boxed{\frac{\pi}{6}}$$

(7) $\int 2^{7x}\,dx \qquad \int a^u = \frac{1}{\ln a} a^u + C$

$u = 7x$
$du = 7\,dx \qquad \frac{1}{7}\int a^u\,du \Rightarrow \frac{1}{7}\left(\frac{1}{\ln 2}\,2^{7x}\right) + C$

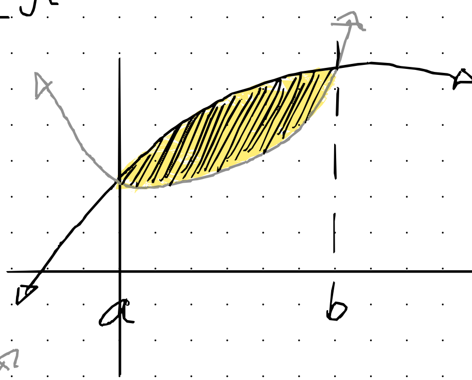
7.1 Area Between Two Curves

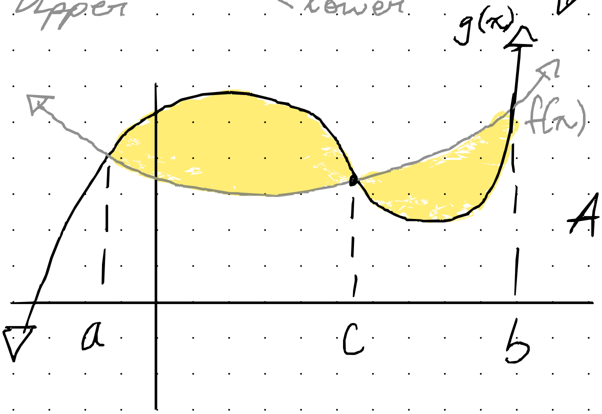
Area between 2 curves:



$$A=\int_a^b \left(f(x)-g(x)\right)\,dx$$

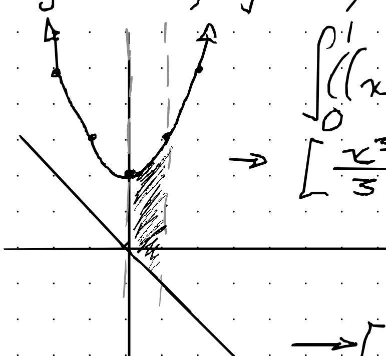
upper   lower





$$A=\int_a^c \left(g(x)-f(x)\right)dx+\int_c^b \left(f(x)-g(x)\right)dx$$

Ex 1: $y=x^2+2,\quad y=-x,\quad x=0,\quad x=1$



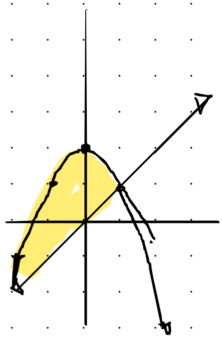
$$\int_0^1 \left((x^2+2)-(-x)\right)dx$$

$$\rightarrow \left[\frac{x^3}{3}+2x+\frac{x^2}{2}\right]_0^1$$

$$\left[\left(\frac{1}{3}+2+\frac{1}{2}\right)-0\right]$$

$$\longrightarrow \left[\frac{2}{6}+\frac{12}{6}+\frac{3}{6}\right] \rightarrow \boxed{\frac{17}{6}}$$

Ex2: $f(x) = 2 - x^2 \quad g(x) = x$



$[-2, 1] \quad f = g$

$2 - x^2 = x$

$0 = x^2 + x - 2 \Rightarrow 0 = (x+2)(x-1)$

$x = -2$

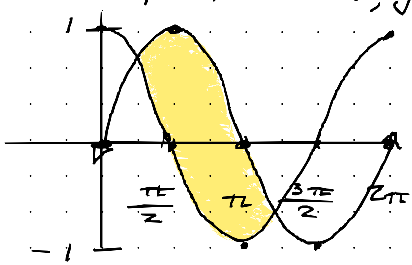
$x = 1$

$A = \int_{-2}^{1} \left((2 - x^2) - (x) \right) dx$

$\rightarrow \boxed{\frac{9}{2}}$ or 4.5

If $[a, b]$ is missing, find the missing x-values through solving for points of intersection

Ex 3: $f(x) = \sin x,\ g(x) = \cos(x)$



$A = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\, dx$

$\rightarrow \boxed{2.828}$

Ex 4: $f(x) = 3x^3 - x^2 - 10x \quad g(x) = -x^2 + 2x$

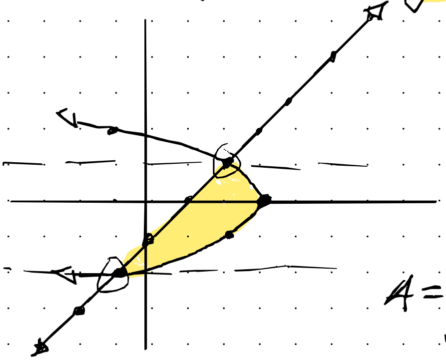
$A = \int_{-2}^{0} (f(x) - g(x))\, dx + \int_{0}^{2} (g(x) - f(x))\, dx$

$12 + 12 \Rightarrow \boxed{24}$

Ex 5: $x = 3 - y^2 \quad x = y + 1$

$\hookrightarrow y = x - 1$

$\hookrightarrow y^2 = 3 - x \Rightarrow y = \pm\sqrt{3 - x}$



$3 - y^2 = y + 1 \Rightarrow 0 = y^2 + y - 2$

$0 = (y+2)(y-1)$

$y = -2 \quad y = 1$

$$A = \int_{-2}^{1} \left((3 - y^2) - (y+1) \right) dy$$

$\longrightarrow \boxed{9/2}$ or 4.5

Note:

The "a" & "b" limits come from "y" axis in horizontal problems, not the "x". Use "dy". Solve the points of intersections for "y". Reimagine the integral in terms of "x" to think of it easier. The upper function is to the right.

7.2 Disk & Washer Method

7.2 Disk and Washer Method

$x^2+8x+12 = -x-2 \to 0 = x^2+9x+14$

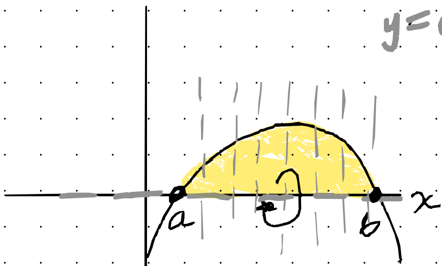
$A = \int_{-3}^{-2} ((-x-2)-(x^2+8x+12))\,dx + \int_{-2}^{-1} ((x^2+8x+12)+x+2)\,dx$

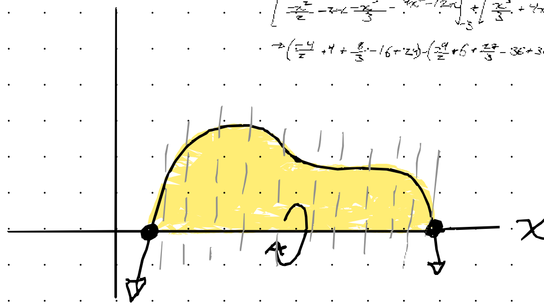
$\int_{-3}^{-2} -x-2-x^2-8x-12\,dx \qquad \int_{-2}^{-1} x^2+8x+12+x+2$

$\left[-\frac{x^2}{2}-2x-\frac{x^3}{3}-4x^2-12x\right]_{-3}^{-2} + \left[\frac{x^3}{3}+4x^2+12x+\frac{x^2}{2}+2x\right]_{-2}^{-1}$

$\to \left(\frac{-4}{2}+4+\frac{8}{3}-16+24\right)-\left(\frac{-9}{2}+6+\frac{27}{3}-36+36\right)$

Solids of Revolution Intro



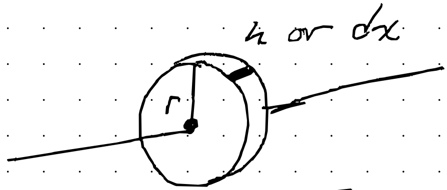


→ Forms a solid object through revolutions

→ Then slice into discs to integrate

V Cylinder/disc $= \underbrace{\boxed{\pi r^2}}_{}h$ (= Area of base) $\to \Delta x$ or $\frac{d}{dx}$

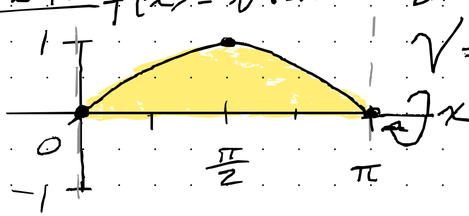
$\int_a^b \pi r^2\,dx$



$V = \pi\int_a^b (f(x))^2\,dx$

$\to \boxed{\pi\int_a^b R(x)^2\,dx}$ Volume of revolvable solid

Ex 1: $f(x) = \sqrt{\sin x}$ and $0 \le x \le \pi$



$V = \pi\int_0^\pi \sqrt{\sin x}^2\,dx$

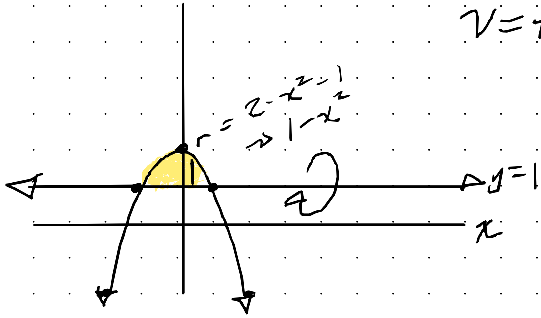
$\to \pi\int_0^\pi \sin x\,dx \approx \boxed{6.283}$

or 2π

Ex 2: $f(x) = 2 - x^2$, $g(x) = 1$, $y = 1$ (revolve about)

$$V = \pi \int_{-1}^{1} (1 - x^2)^2 \, dx$$

$$\approx 3.351$$



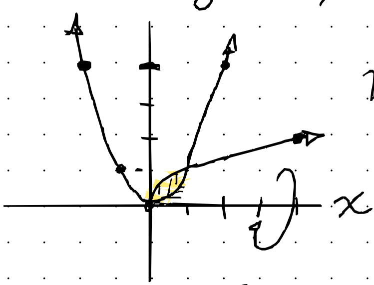
Washer method: $\pi \int_a^b \left(R(x)^2 - r(x)^2 \right) dx$

$R(x)$ = Outermost radius

$r(x)$ = Radius of hole in center

With vertical method: $V = \pi \int_c^d \left([R(y)]^2 - [r(y)]^2 \right) dy$

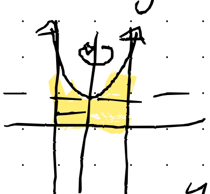
Ex 3: $y = \sqrt{x}$, $y = x^2$, about the x-axis



$$V = \pi \int_0^1 \left((\overset{R}{\sqrt{x}})^2 - (\overset{r}{x^2})^2 \right) dx$$

$$V = \pi \int_0^1 (x - x^4) \, dx \approx \boxed{0.942}$$

Ex 4: $y = x^2 + 1$, $y = 0$, $x = 0$, $x = 1$ about y-axis



$$V = \pi \int_0^1 (1)^2 \, dy + \pi \int_1^2 \left([1]^2 - [\sqrt{y-1}]^2 \right) dy$$

$y - 1 = x^2 \rightarrow x = \pm\sqrt{y-1}$ ≈ 4.712

only use +, taking integral

Cross Sections:

Volumes of Solids with Known Cross Sections

1. Taken perpendicular to x-axis:

$$V = \int_a^b A(x)\,dx$$

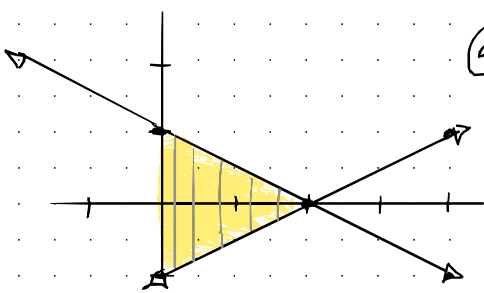
2. Taken perpendicular to y-axis:

$$V = \int_c^d A(y)\,dy$$

Ex 6: Base is bounded by the lines

$f(x) = 1 - \frac{x}{2}$, $g(x) = -1 + \frac{x}{2}$, and $x = 0$

Find the volume. The cross sections are perpendicular to the x-axis. (a) squares (b) semi-circles



(a) $V = \int_0^2 \left(\left(1 - \frac{x}{2}\right) - \left(-1 + \frac{x}{2}\right)\right)^2 dx$

$\approx 2.667 = \boxed{\frac{8}{3}}$

(b) Semi-circle $A = \frac{1}{2}\pi r^2 = \frac{\pi}{2} r^2$

$$V = \int_0^2 \left(\frac{\pi}{2}\left(\frac{(1 - x/2) - (-1 + \frac{x}{2})}{2}\right)^2\right) dx$$

$$= \frac{\pi}{8}\int_0^2 \left(\left(1 - \frac{x}{2}\right) - \left(-1 + \frac{x}{2}\right)\right)^2 dx$$

$$\approx \boxed{1.041}$$